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STABILITY OF PERMANENT ROTATIONS OF A SYMMETRIC SOLID

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We establish that a heavy solid with one fixed point can execute, in the Lagrange case, steady rotations about an axis situated arbitrarily within the body, in addition to a rotation about the dynamic symmetry axis. By combining the integrals of perturbed motion, we find sufficient conditions for the stability of the permanent rotation under consideration. We also indicate the necessary conditions of stability, using the system of the first approximation equations.

The stability of the rotation of a heavy solid with one fixed point about the dynamic symmetry axis situated vertically, was investigated in [1] for the Lagrange case. The necessary and sufficient condition for this rotation in a more general force field was obtained in [2].

Let us consider a solid with one fixed point O, the principal moments of inertia of which are $A = B \neq C$, moving in a field of force which admits a force function $U = U(\gamma_3)$, where γ_3 is the cosine of the angle between the dynamic symmetry axis Oz and the spatially fixed axis Oz_1. The Euler-Poisson equations in this case have the form

$$p^{\cdot} = (1-\delta)\, qr - \gamma_2 u_3, \quad q^{\cdot} = (\delta - 1)\, pr + \gamma_1 u_3, \quad r^{\cdot} = 0 \tag{1}$$
$$\gamma_1^{\cdot} = r\gamma_2 - q\gamma_3, \quad \gamma_2^{\cdot} = p\gamma_3 - r\gamma_1, \quad \gamma_3^{\cdot} = q\gamma_1 - p\gamma_2$$
$$\delta = C / A, \ u_3 = dU / d\gamma_3$$

where p, q and r are the respective projections of the instantaneous angular velocity on the principal axes of inertia Ox, Oy and Oz of the body and γ_1, γ_2 and γ_3 are the direction cosines of the Oz_1-axis in the $Oxyz$ coordinate system.

In addition to the particular solution $p = q = 0$, $r = \omega$, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 = 1$, Eqs.(1) admit the following particular solution just as in the case of a heavy solid [3]

$$p = \omega l_1 = 0, \quad q = \omega l_2, \quad r = \omega l_3, \quad \gamma_1 = l_1 = 0, \quad \gamma_2 = l_2, \quad \gamma_3 = l_3, \quad \omega^2 = \tag{2}$$
$$u_3^\circ / (1-\delta)\, l_3, \quad u_3^\circ = (dU / d\gamma_3)_{\gamma_3 = l_3}$$

Here ω denotes the angular velocity of rotation and the constants l_1, l_2, l_3 are the direction cosines of the Oz_1-axis in the $Oxyz$-axes satisfying the condition $l_1^2 + l_2^2 + l_3^2 = 1$. Choosing $l_1 = 0$ does not affect the generality. In fact we can rotate the x- and y-axes in the equatorial plane of the inertia ellipsoid of the solid in such a way, that the permanent axis Oz_1 is in the same plane as the Oz- and Oy-axis and is orthogonal to the Ox-axis.

The solution (2) of (1) corresponds to the rotation of the solid at a specified angular velocity ω about the Oz_1-axis situated arbitrarily within the body, except when $l_3 = 0$, in which case the angular velocity becomes infinite. The admissible conditions of the problem are represented by the permanent axes for which the quantity ω^2 is positive, and are determined by the inequality $u_3^\circ / (1-\delta)\, l_3 > 0$.

Let us investigate the stability of the motion (2) with respect to the variables q, r, γ_2, q_3 and $q - \omega l_2$, $p - \omega\gamma_1$. We set

$$p = x_1,\ q = \omega l_2 + x_2,\ r = \omega l_3 + x_3,\ \gamma_1 = y_1, \gamma_2 = l_2 + y_2,\ \gamma_3 = l_3 + y_3 \quad (3)$$

in the perturbed motion. The equations of the perturbed motion obtained from (1) with help of (3) admit the following first integrals:

$$V_1 = x_1^2 + x_2^2 + \delta x_3^2 + 2\omega\,(l_2x_2 + \delta l_3x_3) - 2(u_3^\circ y_3 + {}^1/_2 u_{33}^\circ y_3^2) + \ldots = \text{const}, \quad V_2 = x_1y_1 + x_2y_2 + l_2x_2 + \omega l_2y_2 + \delta\,(l_3x_3 + \omega l_3y_3 + x_3y_3) = \text{const}, \quad V_3 = y_1^2 + y_2^2 + y_3^2 + 2\,(l_2y_2 + l_3y_3) = 0, \quad V_4 = x_3 = \text{const}, \quad u_{33}^\circ = (d^2U \,/\, d\gamma_3^2)_{\gamma_3 = l_3} \quad (4)$$

where the dots denote terms of at least third order in y_3 .

To study the stability of the unperturbed motion (2) with respect to the variables r, γ_3, $p - \omega\gamma_1$ and $q - \omega\gamma_2$, we construct the Liapunov function according to the Chetaev method, in the form of the following quadratic combination of the integrals (4):

$$V = V_1 - 2\omega V_2 + \omega^2 V_3 + \lambda V_4^2 = (x_1 - \omega y_1)^2 + (x_2 - \omega y_2)^2 + (\delta + \lambda)\, x_3^2 - 2\omega\delta x_3y_3 + (\omega^2 - u_{33}^\circ)\, y_3^2 + \ldots \quad (5)$$

The quadratic part of the function (5) is positive-definite in the variables $x_1 - \omega y_1$, $x_2 - \omega y_2$, x_3 and y_3, provided that the inequality

$$(\delta + \lambda)\,(\omega^2 - u_{33}^\circ) - \delta^2\omega^2 > 0$$

holds. This can be made to hold by appropriate choice of the constant λ, provided that the condition

$$\omega^2 - u_{33}^\circ = u_3^\circ \,/\, (1 - \delta)\, l_3 - u_{33}^\circ > 0 \quad (6)$$

holds.

Since under the condition (6) the function (5) is positive-definite for sufficiently small values of y_3 and its derivative is identically equal to zero by virtue of the equations of perturbed motion, therefore the inequality (6) represents the sufficient condition of stability of the unperturbed motion (2) with respect to the variables $p - \omega\gamma_1$, $q - \omega\gamma_2$, r and γ_3 (see Rumiantsev theorem in [4]).

The stability with respect to r follows from the fourth integral of (4), hence the inequality (6) is a sufficient condition of stability with respect to the angle of nutation θ ($\cos\theta = \gamma_3$).

To study the stability of the unperturbed motion (2) with respect to the variables q, r, γ_2, γ_3 and $p - \omega\gamma_1$ we choose the Liapunov function in the form

$$V = V_1 - 2\omega V_2 + \omega^2 V_3 + \lambda V_4^2 + \omega x_2 V_3 \,/\, l_2 = (x_1 - \omega y_1)^2 + x_2^2 + \omega^2 y_2^2 + (\delta + \lambda)\, x_3^2 - 2\omega\delta x_3y_3 + (\omega^2 - u_{33}^\circ)\, y_3^2 + 2\omega l_3x_2y_3 \,/\, l_2 + \ldots \quad (7)$$

Function (7) is positive-definite in sufficiently close neighborhood of the coordinate origin of the x_i, y_i variable space, if its quadratic part is positive-definite. The latter takes place when the condition

$$(\delta + \lambda)\,[\omega^2\,(1 - l_3^2 \,/\, l_2^2) - u_{33}^\circ] - \delta^2\omega^2 > 0$$

holds. This in turn is true for sufficiently large values of λ, provided that the inequality

$$\omega^2\,(1 - l_3^2 \,/\, l_2^2) - u_{33}^\circ = u_3^\circ\,(1 - l_3^2 \,/\, l_2^2) \,/\, (1 - \delta)\, l_3 - u_{33}^\circ > 0 \quad (8)$$

holds.

The derivative of (7) has, by virtue of the equations of perturbed motion, the form $V^{\cdot} = \omega x_2^{\cdot} V_3 / l_2 = 0$, and this is correct since $V_3 = 0$. Therefore, on the basis of the Rumiantsev theorem [4] the inequality (8) is a sufficient condition of stability of the unperturbed motion (2) with respect to the variables $p - \omega\gamma_1$, q, r, γ_2 and γ_3.

The unstable permanent rotations (2) can be separated out by considering the linearized system of equations of perturbed motion

$$x_1^{\cdot} = (1 - \delta)\,\omega\,(l_3 x_2 + l_2 x_3) - u_3^\circ y_2 - l_2 u_{33}^\circ y_3, \quad x_2^{\cdot} = (\delta - 1)\,\omega\,(l_3 x_1 + l_1 x_3) + u_3^\circ y_1, \quad y_1^{\cdot} = - l_3 x_2 + l_2 x_3 + \omega\,(l_3 y_2 - l_2 y_3), \quad y_2^{\cdot} = l_3 x_1 - \omega l_3 y_1, \quad y_3^{\cdot} = - l_2 x_1 + \omega l_2 y_1, \quad x_3^{\cdot} = 0 \tag{9}$$

The characteristic equation of (9) has the form

$$\sigma^4 (\sigma^2 + g_0) = 0, \;\; g_0 = \omega^2 [1 + (1 - \delta)^2\, l_3^2] - u_{33}^\circ l_2^\circ + 2u_3^\circ l_3 \tag{10}$$

It is clear that when $g_0 < 0$, one of the roots of (10) is positive and the motion (2) in its first approximation will, by the Liapunov theorem on stability, be unstable.

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CERTAIN PARTICULAR CASES OF STABILITY IN FIRST APPROXIMATION OF DIFFERENCE SYSTEMS

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The results of this paper can be regarded as a transposition of the results of Chetaev obtained for the finite systems of differential equations [1] to the denumerable systems of the finite difference equations. We use the concepts of [2].

Let us consider the system

$$y_s(m + 1) = \sum_{i=1}^{\infty} p_{si}(m)\, y_i(m), \quad m = 0, 1, \ldots \tag{1}$$

Here and henceforth $s = 1, 2, \ldots$, the functions p_{si} are bounded and the series $| p_{s1}(m) | + | p_{s2}(m) | + \ldots$ converge uniformly in m for $0 \leqslant m < \infty$. We define $\| y(m) \| = \sup_s | y_s(m) |$.